Fast and accurate Fresnel scalar diffraction for starshade modeling

Alex Barnett[1]

Starshade Science and Industry Partnership, 3/25/21

Thanks for discussion: David Spergel, Stuart Shaklan, Anthony Harness, Philip Dumont

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Overview

Starshade design needs fast Fresnel diffraction from hard-edged ("0-1") occulter, evaluated at many (eg 10^6) target-plane (pupil) points

- New method: has speed of FFT, but high accuracy of edge-integral
- $10^4\times$ faster than edge-integral methods in starshade context

Today:

- Explain how works. Ingredients:
  - i) areal quadrature: *accurate* rule for 2D integral over occulter
  - ii) "nonuniform FFT": a black-box software library
- Test results, demos

"Efficient high-order accurate Fresnel diffraction via areal quadrature and the nonuniform FFT," A. H. Barnett, *J. Astron. Telesc. Instrum. Syst.* **7**(2), 021211 (21 pages), 2021. `arxiv:2010.05978`.

Code/doc: `https://github.com/ahbarnett/fresnaq`

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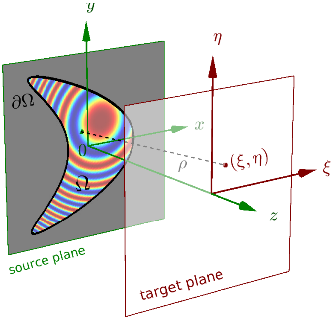
Fresnel scalar diffraction setup and task

Region $\Omega \subset \mathbb{R}^2$ is planar occulter (eg, starshade)

Unit-amplitude incident plane wave along z-axis: target plane field u^{oc}



Babinet: $u^{\text{oc}}(\xi,\eta) \;=\; 1 - u^{\text{ap}}(\xi,\eta)$,

$$u^{\text{ap}}(\xi,\eta) = \frac{1}{i\lambda z}\iint_\Omega e^{\frac{i\pi}{\lambda z}\left[(\xi-x)^2+(\eta-y)^2\right]}\,dxdy$$

λ = wavelength $\quad$ z = downstream dist.

"0-1" source func. convolved w/ complex Gaussian

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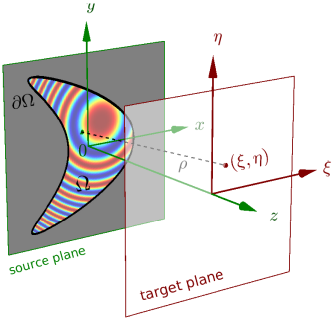

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Fresnel number $\mathfrak{f} := \dfrac{R^2}{\lambda z} \sim 5$ to 20 for starshades $\quad R$ = max radius

Is Fresnel approx. good? yes! next term $\frac{R^4}{\lambda z^3} \sim 10^{-7}$ $\quad$ even for scale models

Is scalar approx. good? yes $\quad$ for full scale; not perfect for scale models

Need u^{oc} abs error $< 10^{-6}$ $\quad$ to model intensity suppression 10^{-10}

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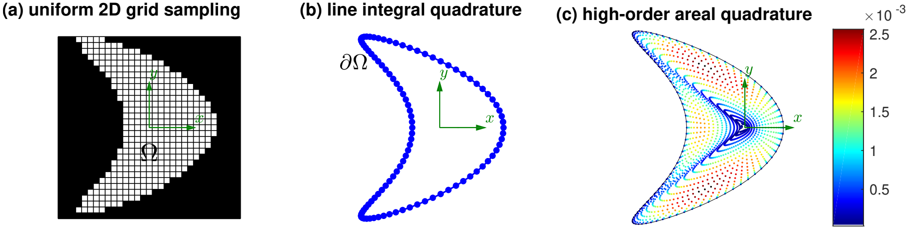
Overview numerical approaches

Two usual approaches for apertures/occulters: we propose a third. . .



2D FFT (or pair)

convolution

fast $\mathcal{O}(n^2 \log n)$

(Mas, Lo, Junchang et al)

low-order $\mathcal{O}(1/n)$

sub-pixel at best $\mathcal{O}(1/n^2)$

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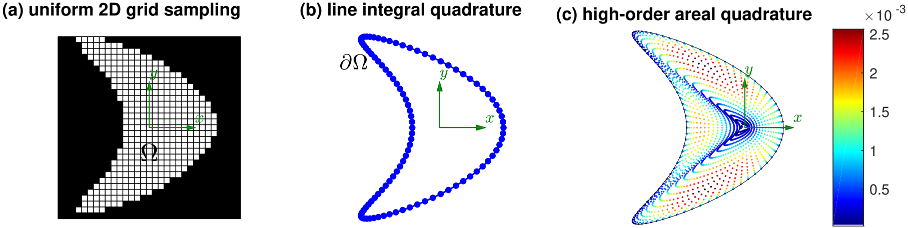

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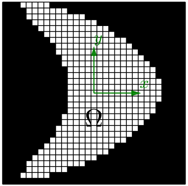

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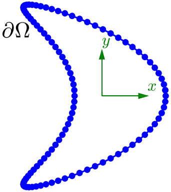

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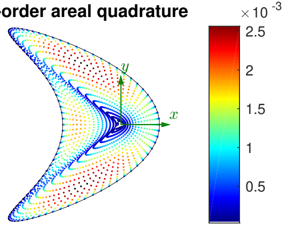

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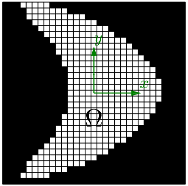

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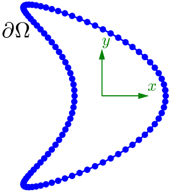

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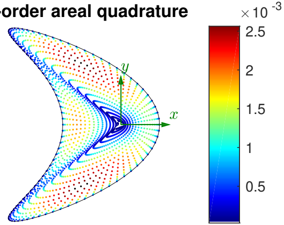

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high-order accurate

Define "high-order": error $\leq C/n^p$ for large algebraic order p (eg 10)

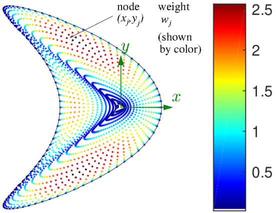
or error $\leq Ce^{-\alpha n}$ exponential

n = linear resolution

Areal quadrature over Ω

AQ is simply set of nodes (x_j, y_j), $j = 1, \ldots, N$, with weights w_j, so

$$\iint_{\Omega} f(x,y)\,dxdy \approx \sum_{j=1}^{N} f(x_j, y_j) w_j$$



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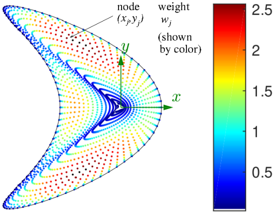

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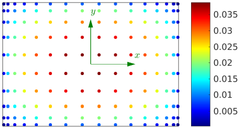
Eg: $\Omega =$ rectangle:

product Gauss–Legendre rule



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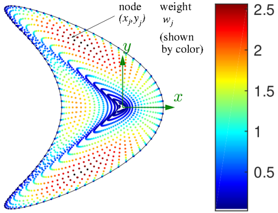

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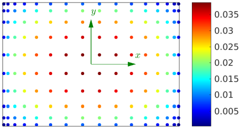


AQ for "arbitrary" geometries?

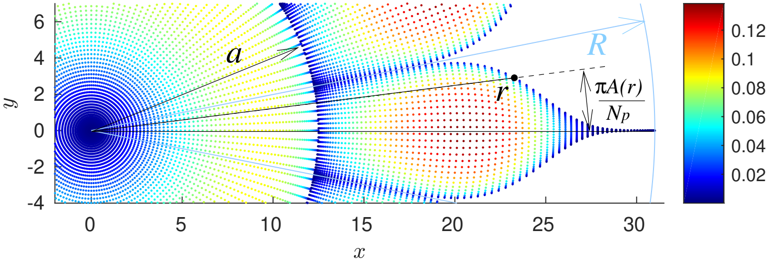
- get from existing line (edge) integral quadrature rule, via dilation
- union of simple pieces + smooth transformations — Jacobean scales w_j
- auto-generate from CAD/FEM formats? probably, not attempted

need precise ($< 10^{-6}$) geometry description; what format?

Areal quadrature for starshades

Easy to build AQ for ideal starshade $N \sim 10^4$–10^6, err 10^{-6}, 20 lines MATLAB

High-order interpolation from points giving petal apodization func $A(r)$:



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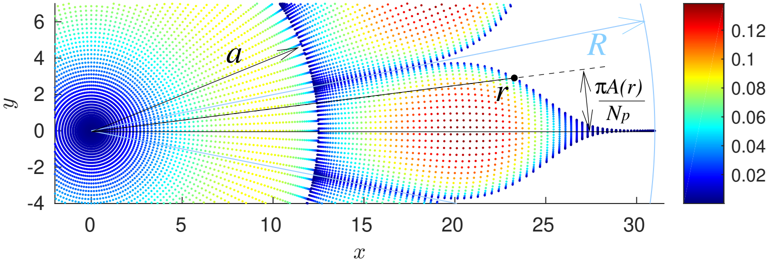

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Non-ideal starshades:

- get from existing line (edge) integral quadrature rule, via dilation

  fine, but not very efficient, $N \sim 10^8$ too large

- adding/subtracting a variable-width strip region to an ideal one?
- rigid motion of petals + AQs for gluing pieces? Add/sub defects. . .

We need to talk: *precise* geometry description, noise autocorr. . .

The fast algorithm: factorization

For all targets $k = 1, \ldots, M$, eval. quadrature rule for Frensel integral:

$$u_k^{\text{ap}} \approx \frac{1}{i\lambda z} \sum_{j=1}^{N} e^{\frac{i\pi}{\lambda z}\left[(\xi_k - x_j)^2 + (\eta_k - y_j)^2\right]} w_j$$

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$\nearrow$ iii) post-multiply $\uparrow$ ii) 2D "type 3 NUFFT" $\nwarrow$ i) pre-multiply

Three sequential steps (very simple, core $<$ 10 lines of MATLAB)

Cost $\mathcal{O}(N + M + \mathfrak{f}^2 \log \mathfrak{f})$ In practice: 10^7 targets/sec on laptop

If targets on regular grid, use (faster) type 1 NUFFT Abbrev by "t3" and "t1"

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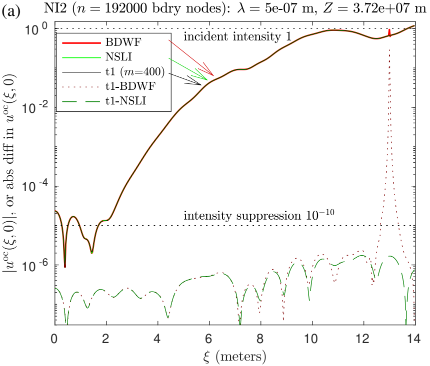
Recommended software library for NUFFTs: `FINUFFT`

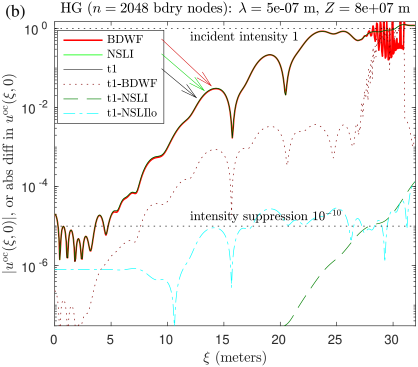
`http://finufft.readthedocs.io`

C++/OpenMP; beats others by $10\times$ (Barnett et al '19)

Performance & validation for ideal starshades NI2, HG

Validation vs edge-integrals: (Cady '12; BDWF as pulled from SISTER codebase)





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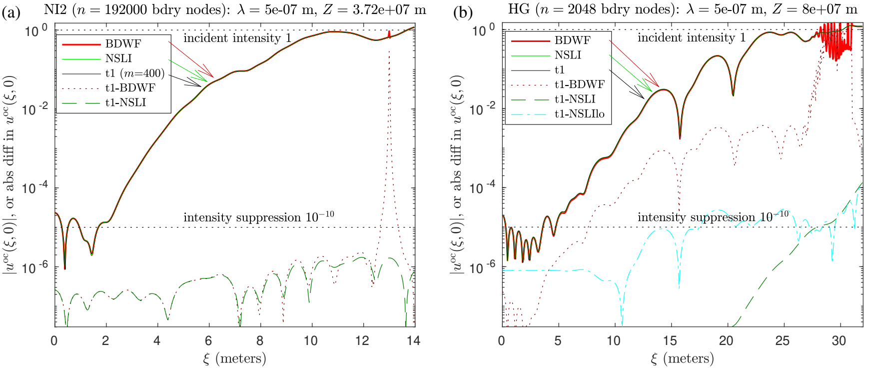


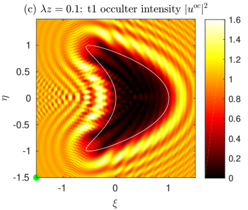
Speed against BDWF, for million-point target grid: i7 laptop

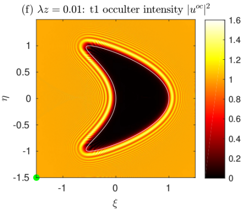
design	λ (m)	z (m)	$\mathfrak{f}$	m (petal)	total nodes	M (targets)	method	CPU time
NI2	5e-7	3.72e7	9.1	6000	n=192000	10^6, grid	BDWF	5361 s
				400	N=499200		NUFFT t1 (ε=10^{-8})	0.076 s
HG	5e-7	8e7	24	60	n=2048	10^6, grid	BDWF	80.5 s
				60	N=37440		NUFFT t1 (ε=10^{-8})	0.042 s

Conclusions: same accuracy reached, 3–5 orders of magnitude faster

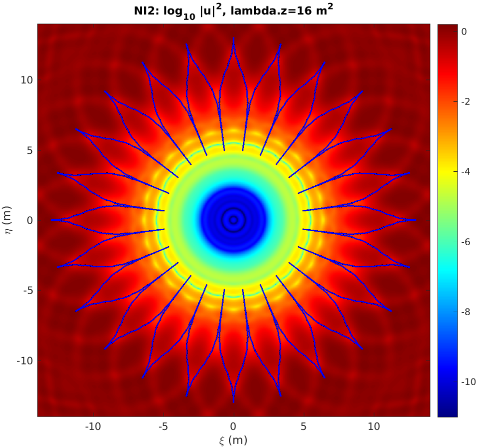
Results pictures

Smooth kite test:





NI2 ideal starshade:



Wavelength sweeps & SISTER modifications

Shadow depth study, 50 λ values, takes 6 seconds: including AQ gen; laptop

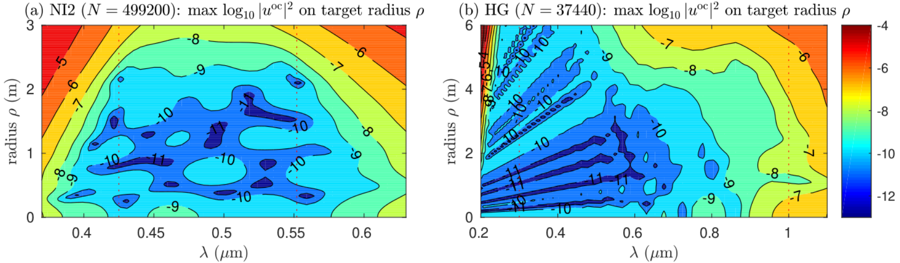


Fig 6 Intensity (on $\log_{10}$ scale indicated on the right) as a function of wavelength and target radius ρ from the center, for the two starshade designs (NI2 and HG) of Fig. 5. At each of 200 ρ values, the maximum over 300 angles is taken. The indicent intensity is 1. The NUFFT t3 method is used. Vertical dotted lines show the designed λ range.

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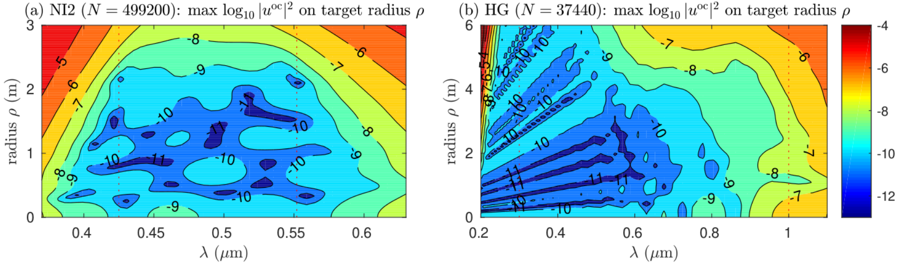

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Replacing BDWF by t3 in SISTER "PSF basis" task: hack, proof-of-principle

- 3149 shifts of 16×16 pupil grids $\rightarrow$ group targets together!
- reduces run-time from 6.5 hours to 2.6 seconds

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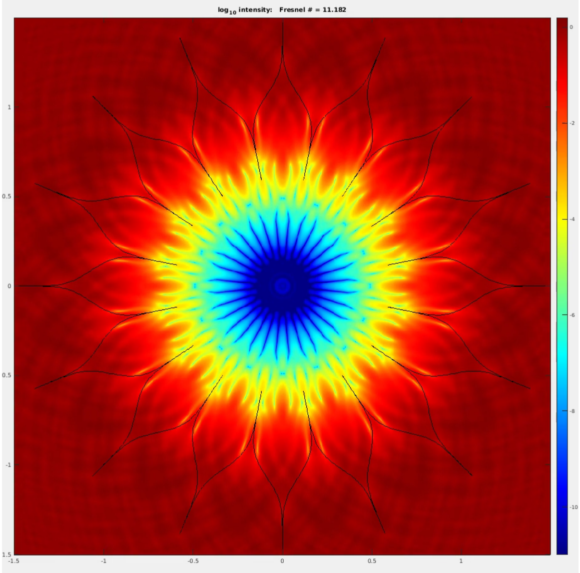
Movie (analytic starshade design)

Computed at around 10 frames/sec (close to real time):



Fun demo of complicated geometry

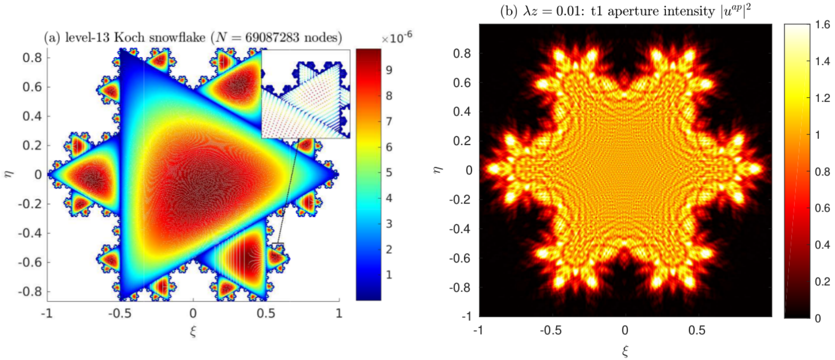


Fig 7 Koch fractal aperture diffraction example from Sec. 3.3. (a) shows the areal quadrature constructed by a union of about 67 million triangles. The color of each node (x_j, y_j) indicates its weight w_j using the scale on the right. The inset shows a zoom into the region shown, resolving individual nodes. (b) shows intensity (on $\log_{10}$ scale indicated on the right) computed on a million-point grid by the NUFFT t1 method in under 5 seconds.

Conclusions

Much faster method for accurate ($< 10^{-6}$) hard-edged Fresnel diffraction

$\sim 10^7$ pupil plane targets/sec, almost indep of starshade geom

excels for large # targets. Crossover vs edge-integral for NI2 starshade: ~ 50 targets

- needs good (high-order) *areal quadrature* for occulter shape
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Future / questions:

- non-ideal (notches, shape variation) started (Dumont, Shaklan)
- continuous phase variation? trivial to include
- probably useful for other "0-1" design problems: coronagraphs, etc

Code/docs: MATLAB, including all figures from paper, and SISTER modifications

`https://github.com/ahbarnett/fresnaq`

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