The Hotelling Observer in Adaptive Optics

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Outline

- Brief review
- Application to AO
- Computational methods
- Conclusions

Brief review

The Hotelling observer:

- Optimal linear discriminant
- Equivalent to ideal observer for Gaussian data
- Requires knowledge of data mean and covariance
- Inversion of a large covariance matrix (usually) required
- Given by:

$$t_{\mathrm{Hot}}(\mathbf{g}) = \mathbf{w}^{\mathsf{T}} \mathbf{g},$$

  for an appropriate "template" vector $\mathbf{w}$
- Maximizes:

$$\mathsf{SNR} = \frac{\langle t_{\mathrm{Hot}}(\mathbf{g}) \rangle_{\mathbf{g}|S_+} - \langle t_{\mathrm{Hot}}(\mathbf{g}) \rangle_{\mathbf{g}|S_-}}{\sqrt{\frac{1}{2}\sigma^2_{t_{\mathrm{Hot}}(\mathbf{g})|S_+} + \frac{1}{2}\sigma^2_{t_{\mathrm{Hot}}(\mathbf{g})|S_-}}}$$

Brief review

- Noise-free image when the signal is absent: $\overline{\mathbf{g}}_0 = \langle \mathbf{g} \rangle_{S_-}$
- Noise-free image when the signal is present: $\overline{\mathbf{g}}_1 = \langle \mathbf{g} \rangle_{S_+}$
- Then the signal (ex: a planet) is $\mathbf{s} = \overline{\mathbf{g}}_1 - \overline{\mathbf{g}}_0$
- Let $\mathbf{K_g}$ be the covariance matrix of the data:

$$\mathbf{K_g} = \Pr(S_-)\mathbf{K}_{\mathbf{g}|S_-} + \Pr(S_+)\mathbf{K}_{\mathbf{g}|S_-}$$

- The template is:

$$\mathbf{w} = \mathbf{K_g}^{-1}\mathbf{s}$$

Application to AO

- For the case of AO images:
- $\mathbf{G} = (\mathbf{g}^{(1)}, \ldots, \mathbf{g}^{(J)})$: sequence of noisy random images
- $\mathbf{P} = (\mathbf{p}^{(1)}, \ldots, \mathbf{p}^{(J)})$: sequence of random residual PSF's
- $\mathbf{F} = (\mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(J)})$: random and time-dependent objects
- Assume $\mathbf{f}^{(1)} = \cdots = \mathbf{f}^{(J)} = \mathbf{f}$ (constant unknown object)
- $\mathbf{f}$ can either be $\mathbf{f}_{S_-}$ (signal absent) or $\mathbf{f}_{S_+}$ (signal present)
- Two sources of randomness:
  - Residual random PSF
  - Measurement noise: Poisson noise, readout noise

Application to AO

- If we want to use the Hotelling observer in AO, we need to find the mean image sequences under both hypotheses and the mean data covariance matrix
- For the mean image sequence given $\mathbf{f}_i$ ($i = S_-$ or $i = S_+$):

$$\overline{\overline{\mathbf{G}}}_i = \left\langle \left\langle \mathbf{G} \right\rangle_{\mathbf{G}|\mathbf{P},\mathbf{f}_i} \right\rangle_{\mathbf{P}|\mathbf{f}_i}$$

- Likewise:

$$\mathbf{K}_{\mathbf{G}|i} = \left\langle \left\langle \left[\mathbf{G} - \overline{\overline{\mathbf{G}}}_i\right] \left[\mathbf{G} - \overline{\overline{\mathbf{G}}}_i\right]^{\mathsf{T}} \right\rangle_{\mathbf{G}|\mathbf{P},\mathbf{f}_i} \right\rangle_{\mathbf{P}|\mathbf{f}_i}$$

- Dim signal: $\mathbf{K}_{\mathbf{G}|S_-} \approx \mathbf{K}_{\mathbf{G}|S_+} \approx \mathbf{K}_{\mathbf{G}}$
- It can be proved that:

$$\mathbf{K}_{\mathbf{G}} = \overline{\mathbf{K}}_{\mathbf{G}}^{\text{noise}} + \mathbf{K}_{\overline{\mathbf{G}}}^{\text{PSF}}$$

Application to AO

- $\overline{\overline{\mathbf{G}}}_i$ estimated from simulated data
- Generate L realizations of AO-corrected system function

$$\left[\overline{\overline{\mathbf{G}}}_i\right]_m^{(j)} = \int_{\mathbb{R}^2} \overline{h}_m^{(j)}(x, y)\mathbf{f}_i(x, y)\,\mathrm{d}x\,\mathrm{d}y,$$

where:

$$\overline{h}_m^{(j)}(x, y) = \frac{1}{L}\sum_{\ell=1}^{L} h_{\ell,m}^{(j)}(x, y)$$

Application to AO

- $\mathbf{K}_{\overline{\mathbf{G}}}^{\mathsf{PSF}}$ estimated from simulated data:

$$\left[\mathbf{K}_{\overline{\mathbf{G}}}^{\mathsf{PSF}}\right]_{m,m'}^{(j,j')} = \frac{1}{L-1} \sum_{\ell=1}^{L} \Delta h_{\ell,m}^{(j)}(x,y)\, \Delta h_{\ell,m'}^{(j')}(x,y),$$

where:

$$\Delta h_{\ell,m}^{(j)}(x,y) = h_{\ell,m}^{(j)}(x,y) - \frac{1}{L} \sum_{\ell'=1}^{L} h_{\ell',m}^{(j)}(x,y)$$

- Formally:

$$\mathbf{W} = \mathbf{K}_{\overline{\mathbf{G}}}^{-1} \left[\overline{\overline{\mathbf{G}}}_{S_+} - \overline{\overline{\mathbf{G}}}_{S_-}\right]$$

Application to AO

- The big problem:

$$\mathbf{W} = \left[\mathbf{K}_{\mathbf{G}} \right]^{-1} \left[\overline{\overline{\mathbf{G}}}_{S_+} - \overline{\overline{\mathbf{G}}}_{S_-} \right]$$

- For example, if images are 64×64 ($M = 64^2$ pixels) and there are $J = 25$ of them in each sequence, then $\mathbf{K}_{\mathbf{G}}$ is 102400×102400
- Storing $\mathbf{K}_{\mathbf{G}}$ as `double`-valued matrix requires about 39 GB of memory!
- And you still need to invert it...

Computational methods

- Matrix-inversion lemma:

$$[\mathbf{N} + \mathbf{R}\mathbf{R}^{\mathsf{T}}]^{-1} = \\ = \mathbf{N}^{-1} - \mathbf{N}^{-1}\mathbf{R}[\mathbf{I}_n + \mathbf{R}^{\mathsf{T}}\mathbf{N}^{-1}\mathbf{R}]^{-1}\mathbf{R}^{\mathsf{T}}\mathbf{N}^{-1}$$

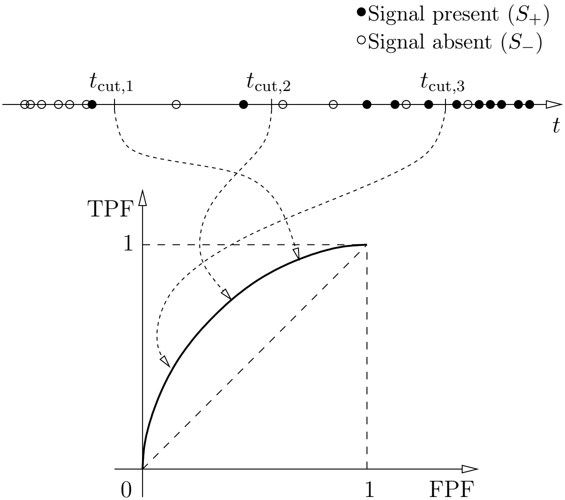
- Recall: $\mathbf{K}_{\mathbf{G}} = \overline{\mathbf{K}}_{\mathbf{G}}^{\text{noise}} + \mathbf{K}_{\overline{\mathbf{G}}}^{\text{PSF}}$
- $\mathbf{K}_{\overline{\mathbf{G}}}^{\text{PSF}}$ estimated from L sample sequences
- $\mathbf{K}_{\overline{\mathbf{G}}}^{\text{PSF}} = \mathbf{R}\mathbf{R}^{\mathsf{T}}$ where $\mathbf{R}$ is $MJ \times L$
- Substituting:

$$\mathbf{K}_{\mathbf{G}}^{-1} = \left[\overline{\mathbf{K}}_{\mathbf{G}}^{\text{noise}}\right]^{-1} + \left[\overline{\mathbf{K}}_{\mathbf{G}}^{\text{noise}}\right]^{-1}\mathbf{R}\mathbf{Q}^{-1}\mathbf{R}^{\mathsf{T}}\left[\overline{\mathbf{K}}_{\mathbf{G}}^{\text{noise}}\right]^{-1}$$

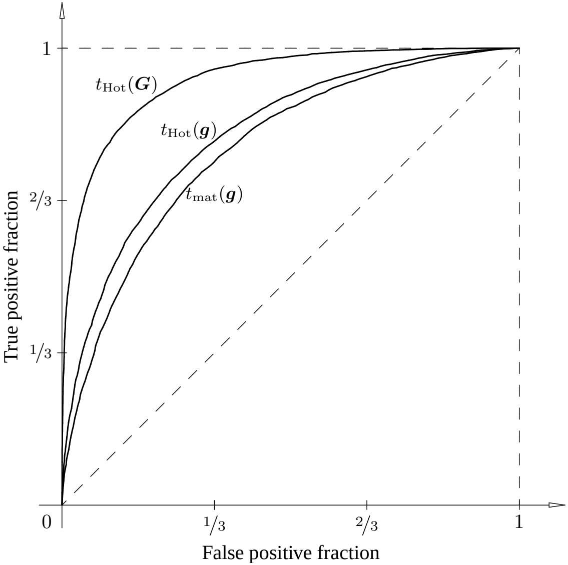
for $\mathbf{Q} = \mathbf{I}_L + \mathbf{R}^{\mathsf{T}}\left[\overline{\mathbf{K}}_{\mathbf{G}}^{\text{noise}}\right]^{-1}\mathbf{R}$

- $\mathbf{Q}$ is $L \times L$ and L does not have to be large

Computational methods

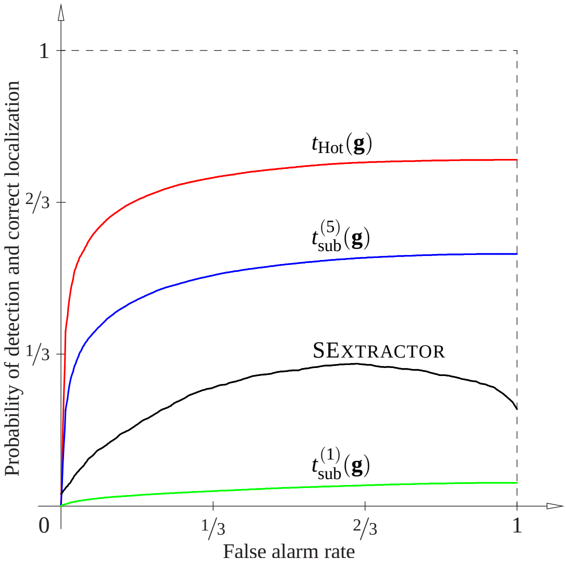


Results



1
2/3
1/3
0
1/3
2/3
1
True positive fraction
False positive fraction
$t_{\mathrm{Hot}}(\boldsymbol{G})$
$t_{\mathrm{Hot}}(\boldsymbol{g})$
$t_{\mathrm{mat}}(\boldsymbol{g})$

Results



1
2/3
1/3
0
1/3
2/3
1
Probability of detection and correct localization
False alarm rate
$t_{\mathrm{Hot}}(\mathbf{g})$
$t_{\mathrm{sub}}^{(5)}(\mathbf{g})$
SEXTRACTOR
$t_{\mathrm{sub}}^{(1)}(\mathbf{g})$

References

- Harrison H. Barrett, Kyle J. Myers, Nicholas Devaney, and Christopher Dainty, "Objective assessment of image quality. IV. Application to adaptive optics," *Journal of the Optical Society of America A*, vol. 23, issue 12, pp. 3080–3105 (2006)
- Luca Caucci, Harrison H. Barrett, and Jeffrey J. Rodríguez, "Spatio-temporal Hotelling observer for signal detection from image sequences," *Optics Express*, vol. 17, issue 13, pp. 10946–10958 (2009)